Math 381 Complex Variables and Transforms

Instructor: Matthias Nagel due in class 13:35, Feb 1

Homework 2

Recall that the derivates of the functions $\sinh\colon \mathbb{R} \to \mathbb{R}$ and $\cosh\colon \mathbb{R} \to \mathbb{R}$ are related by $\sinh'(x) = \cosh(x)$ and $\cosh'(x) = \sinh(x)$.

Exercise 2.1. Simplify the following expression for $\theta = \frac{2\pi}{7}$:

$$1 + \cos(\theta) + \cos(2\theta) + \ldots + \cos(7\theta).$$

Hint: Recall Exercise 1.5.

Exercise 2.2. Calculate the (complex) derviatives of the following functions

1. $f(z) := \frac{z-1}{2z+1}$ for $z \neq \frac{-1}{2}$
2. $f(z) := z^2(1 + z^{-2})^4$ for $z \neq 0$.

Exercise 2.3. Verify that the function $f(x + iy) = \sqrt{|xy|}$ has no complex derivative in 0. Also check that the functions $u(x, y)$ and $v(x, y)$ with $f(x + iy) = u(x, y) + iv(x, y)$ do fulfil the Cauchy-Riemann equations in the point 0.

Exercise 2.4. Do the following functions u, v fulfill the Cauchy-Riemann equations?

1. $u(x, y) = e^{-x} \cos y$ and $v(x, y) := -e^{-x} \sin y$
2. $u(x, y) = \cos x \cosh y$ and $v(x, y) = -\sin x \sinh y$.

Exercise 2.5. Find a harmonic conjugate to the functions

1. $u(x, y) := x^3 - 3xy^2$
2. $u(x, y) := \sinh x \cdot \sin y$
3. $u(x, y) := \frac{y}{x^2+y^2}$

Exercise 2.6. Calculate the line integrals $\int_\gamma f(z)dz$ for:

1. $f(z) := \frac{1}{z^2}$ along the curve $\gamma\colon [0, 1] \to \mathbb{C} \setminus \{0\}$ with $\gamma(t) := \exp(2\pi i t)$.
2. $f(z) := \bar{z}$ along the curve $\gamma\colon [0, 1] \to \mathbb{C}$ with $\gamma(t) := t(1 + i)$.
3. $f(z) := \bar{z}$ along the curve $\gamma\colon [0, 1] \to \mathbb{C}$ with $\gamma(t) := t + it^2$.

Exercise 2.7. Let $f\colon \mathbb{C} \setminus \{i, 1 + i, 1\} \to \mathbb{C}$ be the function defined by

$$f(z) := \frac{1}{z - i} - \frac{i}{z - (1 + i)} + \frac{3}{(z - 1)^2}.$$

Calculate the following line integrals $\int_{\gamma_i} f(z)dz$ for $i = 1, 2, 3$, where the curves γ_i are drawn in Figure 1 below.

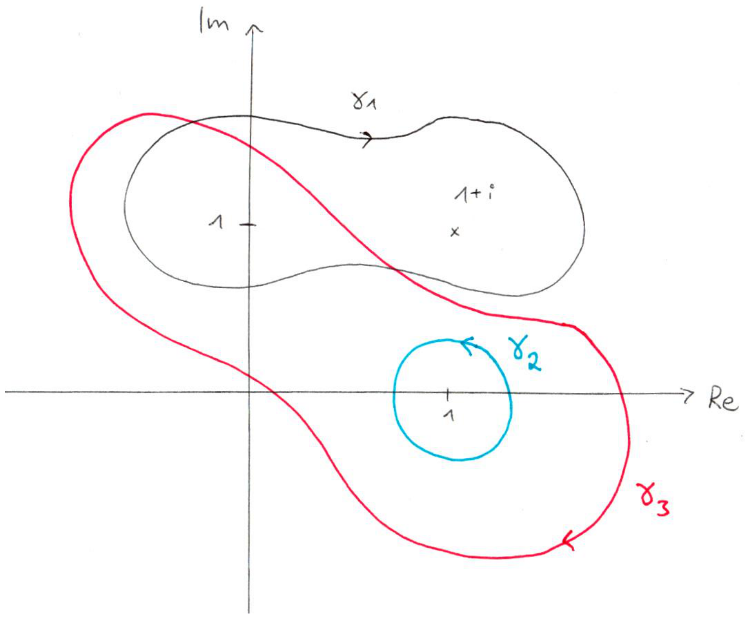

Figure 1: Curves γ_i for Exercise 2.7