Math 731 – Algebraic Topology

Instructor: Matthias Nagel due in class 9:30, October 6

Homework 2

Exercise 2.1 (Homotopic maps)**.** (1) Show that the relation of homotopy equivalence of maps $X \to Y$ is an equivalence relation.

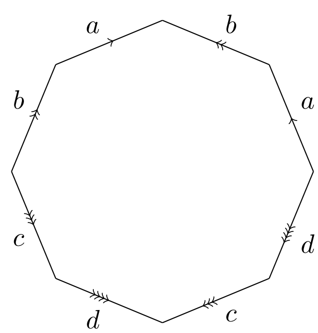
(2) Let $f\colon X \to Y$, $f'\colon X \to Y$, and $g\colon Y \to Z$, and $g'\colon Y \to Z$ such that $f \simeq f'$, and $g \simeq g'$. Then $f \circ g \simeq f' \circ g'$.

(3) Let $f, f'\colon X \to Y$, and $a, b \in X$ with $f(a) = f'(a)$ and $f(b) = f'(b)$. We have defined associated maps

$$\pi_1(f), \pi_1(f')\colon \pi_1(X, a, b) \to \pi_1(Y, f(a), f(b))$$

in the lecture. Show if $f \simeq f'$ rel $\{a, b\}$, then $\pi_1(f) = \pi_1(f')$.

Exercise 2.2. The surface of genus 2 is obtained from the octagon by identifying the sides as follows:



Compute its Euler characteristic.

Exercise 2.3. (1) Show that $D^n \times D^m$ is homeomorphic to the closed disk D^{n+m}.

(2) Let X, Y be finite CW-complexes. Equip the product $X \times Y$ with a CW-structure.

Exercise 2.4. Prove or find a counter-example: A map $f\colon X \to Y$ is a homotopy equalence, if there exists map $g\colon Y \to X$ and $h\colon Y \to X$ such that

$$f \circ g \simeq \mathrm{id}_Y \qquad h \circ f \simeq \mathrm{id}_X\,.$$

Exercise 2.5. Denote the set of sequences with values in $\mathbb{R}$ by $\mathbb{R}^{\mathbb{N}}$. Consider the normed vector space

$$\mathbb{R}^\infty = \left\{ x = (x_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \colon \text{only finitely many } x_i \neq 0 \right\}$$

with norm $\|x\| = \sum_i |x_i|$. Denote the unit sphere of $\mathbb{R}^\infty$ by

$$S(\mathbb{R}^\infty) = \left\{ x \in \mathbb{R}^\infty \colon \|x\| = 1 \right\}.$$

(1) Show that the shift map $T\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by
$$T(x)_i = \begin{cases} 0 & i = 0 \\ x_{i-1} & i > 0 \end{cases},$$
is continuous and restricts to a map $T\colon S(\mathbb{R}^\infty) \to S(\mathbb{R}^\infty)$. Show that this restriction is homotopic to the identity $\mathrm{id}_{S(\mathbb{R}^\infty)}$.

(2) Show that $T \simeq c_{(1,0,\ldots)}$, and deduce that $S(\mathbb{R}^\infty)$ is contractible.