Math 731 – Algebraic Topology

Instructor: Matthias Nagel — due in class 9:30, November 17

Homework 4

Exercise 4.1. Let G be a group with a free generating system $S \subset G$, and G' be a group with a free generating system $S' \subset G'$. Suppose S and S' have the same cardinality, prove that G is isomorphic to G'. Hint: Use the universeral property of G and G' to obtain homomorphisms between these groups, and use uniqueness to show their composition is the identity.

Exercise 4.2.
(1) Show that $\mathbb{Z}^2$ is not a free group.
(2) Show that the coproduct (free product) $\mathbb{Z}/2 * \mathbb{Z}/3$ of two cyclic groups of order 2 and 3 is not isomorphic to their product $\mathbb{Z}/2 \times \mathbb{Z}/3$.

Exercise 4.3. Can there be a continous action of the integers $\mathbb{Z}$ on S^2 such that for all $x \in S^2$ there exists an open subset $U_x \ni x$ with

$$U_x \cap gU_x = \emptyset \text{ for every } g \neq e.$$

Exercise 4.4. Let $f\colon (X, x_0) \to (Y, y_0)$ be a based map of path-connected spaces.
(1) if f is surjective, does $\pi_1(f)$ have to be surjective?
(2) if f is injective, does $\pi_1(f)$ have to be injective?