Introduction to 3–manifolds

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Exercise 1.1.

1. Using the stereographic projection, show that S^3 is homeomorphic to the one-point compactification $\mathbb{R}^3 \cup \{\infty\}$.
2. The set $L = S^1 \times \{0\} \sqcup \{0\} \times S^1 \subset \mathbb{C}^2$ happens to be a submanifold of
$$S^3 = \big\{(x, y) \in \mathbb{C}^2 \colon \|(x, y)\| = 1\big\}.$$
Determine (and visualise) its image under the stereographic projection.

Exercise 1.2. Let Σ be a compact surface and $\varphi \in \operatorname{Diff}(\Sigma)$ a diffeomorphism. Define an action by $\mathbb{Z}$ on $\mathbb{R} \times \Sigma$ via

$$k \cdot (t, x) = \big(t + k, \varphi^{-k}(x)\big).$$

1. Show that this action is a covering action.
2. Construct a 2–fold covering map $M_{\varphi^2} \to M_\varphi$ between the mapping tori.

Exercise 1.3. Let G be a group acting freely on a manifold M. Consider the quotient map $\pi \colon M \to N$ with $N = M/G$. Suppose that N is hausdorff and π a local homeomorphism. Deduce that the action by G is a covering action.

Exercise 1.4. Let M be an n–manifold and X a k–manifold. Let $f \colon X \to M$ be a proper embedding. Prove that $f(X)$ is a neat submanifold, if $f(\partial X) \subset \partial M$, $f(\operatorname{Int} X) \subset \operatorname{Int} M$ and $(T_x f)(T_x X) \not\subset T_{f(x)}\partial M$ for all $x \in \partial X$.

Exercise 1.5. Prove that if a point p of a manifold M is mapped to $\partial\mathbb{R}^n_+$ by one boundary chart, it is mapped to $\partial\mathbb{R}^n_+$ by any (boundary) chart.
Hint: Use that $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}) \cong \mathbb{Z}$, but $H_n(\mathbb{R}^n_+, \mathbb{R}^n_+ \setminus \{0\}; \mathbb{Z}) \cong 0$.