Introduction to 3–manifolds

Matthias Nagel

Exercise 4.1. Show that any embedded 2–torus $T^2 \subset S^3$ separates, and it bounds a solid torus $S^1 \times D^2$ on one of its sides.

Hint: Likely, you need the following fact about amalgamations from group theory: suppose G is the coproduct of A and B along H, that is, G fits in a push-out diagram below

$$\begin{tikzcd} A \arrow[r, "j_A"] & G \\ H \arrow[u, "i_B"] \arrow[r, "i_A"'] \arrow[ur, phantom, "\llcorner" very near end] & B \arrow[u, "j_B"'] \end{tikzcd} .$$

Then, if both i_A and i_B are injective, then so are j_A and j_B.

Exercise 4.2. Let P be the sphere bundle over S^1 with non-orientable total space, which is the mapping torus $M_\varphi(S^2)$ for an orientation-reversing diffeomorphism $\varphi \in \mathrm{Diff}(S^2)$. Show that $P \# P \cong \big(S^1 \times S^2\big) \# P$.

Hint: Find an orientation-preserving loop intersecting a fibre of P in a single point. Now remove neighbourhoods. To determine the complement, fill the neighbourhood of the arc (and two 3–balls) back in.

Exercise 4.3 (Poincaré homology sphere)**.** Consider the *quaternions* $\mathbb{H} = \mathbb{R}\langle 1, i, j, k\rangle$, the 4–dimensional associative $\mathbb{R}$–algebra with $i^2 = j^2 = k^2 = ijk = -1$. We can write any element $z = a \cdot 1 + b \cdot i + c \cdot j + d \cdot k$ for $a, b, c, d \in \mathbb{R}$. Define an involution $z \mapsto \overline{z}$ on $\mathbb{H}$ by

$$a \cdot 1 + b \cdot i + c \cdot j + d \cdot k \mapsto a \cdot 1 - b \cdot i - c \cdot j - d \cdot k,$$

that is $\overline{\overline{z}} = z$ and $\overline{z_0 \cdot z_1} = \overline{z_1} \cdot \overline{z_0}$. Since $z \cdot \overline{z} = a^2 + b^2 + c^2 + d^2$, we obtain a norm on $\mathbb{H}$ defined by $\|z\| = \sqrt{z \cdot \overline{z}}$. The quaternions decompose $\mathbb{H} = \mathbb{R}\langle 1\rangle \oplus \mathbb{R}\langle i, j, k\rangle$ into the eigenspaces of the involution, the *real* and *imaginary quaternions*. The imaginary quaternions $\operatorname{Im}\mathbb{H} := \{z \in \mathbb{H} \colon a = 0\} = \mathbb{R}\langle i, j, k\rangle$ form a 3–dimensional $\mathbb{R}$–vector space.

Now establish the following, where points 1. and 2. have only been added for the sake of completeness. Only prove them, if they align with your interests.

1. Show that conjugation $(q, z) \mapsto qz\overline{q}$ leaves $\operatorname{Im}\mathbb{H}$ invariant, so it restricts to a map $\mathbb{H} \times \operatorname{Im}\mathbb{H} \to \operatorname{Im}\mathbb{H}$, and defines a homomorphism $\mathbb{H} \to \mathrm{Hom}_{\mathbb{R}}(\operatorname{Im}\mathbb{H}, \operatorname{Im}\mathbb{H})$.

2. Show that this map restricts to a homomorphism $\rho \colon S(\mathbb{H}) \to \mathrm{O}(\operatorname{Im}\mathbb{H})$, where $S(\mathbb{H})$ denotes the subgroup $\{z \in \mathbb{H} \colon \|z\| = 1\}$ and $\mathrm{O}(\operatorname{Im}\mathbb{H}) = \{f \in \mathrm{GL}(\operatorname{Im}\mathbb{H}) \colon \|f(v)\| = \|v\| \text{ for } v \in \operatorname{Im}\mathbb{H}\}$.

   Show that ρ is a local diffeomorphism, and that it has kernel $\ker \rho = \{\pm 1\}$. Deduce that it image is an open subgroup, and so it is the identity component $\mathrm{SO}(\operatorname{Im}\mathbb{H})$ of $\mathrm{O}(\operatorname{Im}\mathbb{H})$. Deduce that ρ is a 2–fold cover.

Note: $S(\mathbb{H}) \cong S^3$ and thus, we have constructed the universal cover $\mathrm{Spin}(3) := S(\mathbb{H})$ of $\mathrm{SO}(3)$.

3. Consider the icosahedral group $A \subset \mathrm{SO}(3)$ of orientation-preserving symmetries of the regular icosahedron, which is the alternating group $A \cong A_5$. This group is perfect, i.e. it is equal to its commutator subgroup $A = [A, A]$. Consider its preimage $I_2 = \rho^{-1}(A)$, the *binary icosahedral group.*

   Show that I_2 contains a non-trivial perfect subgroup Γ.[1] Note that $S(\mathbb{H})$ acts on itself, and so $S(\mathbb{H}) \subset O(\mathbb{H}) = O(4)$.

   Show that the action of Γ on $S(\mathbb{H})$ is a covering action, and thus $M := S(\mathbb{H})/\Gamma$ is closed elliptic 3–manifold. The manifold $S(\mathbb{H})/I_2$ is called the *Poincaré homology sphere.*

4. Show that $H_k(M; \mathbb{Z}) \cong H_k(S^3; \mathbb{Z})$ for all $k \geq 0$, but M is not diffeomorphic to S^3.

[1] The group I_2 is actually perfect itself and so we can pick $\Gamma = I_2$, but I am unable to prove this without a concrete description of $A \subset \mathrm{SO}(3)$. Any ideas?