Math 731 – Algebraic Topology

Instructor: Matthias Nagel — due in class 9:30, September 22

Homework 1

Exercise 1.1 (Continuity)**.** Let $(X,d),(Y,d)$ be two metric spaces. Let $f\colon X\to Y$ be a function. Show the following are equivalent:

(1) f is continuous in the ε-δ-sense.
(2) for all $x\in X$ and an open set $V\ni f(x)$, there exists an open set $U\ni x$ such that $f(U)\subset V$.
(3) for all open sets $V\subset Y$, the preimage $f^{-1}(V)$ is open.

Exercise 1.2 (Subspace topology)**.** (1) Let $(X,\mathcal{O})$, $(Y,\mathcal{S})$ be two topological space, and a continuous function $f\colon X\to Y$. Let $A\subset X$ and $B\subset Y$ two subsets such that $f(A)\subset B$. Show the restriction $f|_A\colon A\to B$ is continuous with respect to the subspace toplogies on A and B.
(2) Let $(X,\mathcal{O})$ be a topological space and A a subset of X. Let τ be a topology with the following property: for every topological spaces Y and every continuous map $f\colon X\to Y$, also the restriction $f|_A\colon A\to Y$ is continuous. Show that the subspace topology $\mathcal{O}_A$ is coarser than τ; i.e. $\mathcal{O}_A\subset\tau$.

Exercise 1.3. (1) Show that S^n is path-connected for $n\geq 1$. Hint: you can use the stereographic projection.
(2) Show that the upper hemisphere $D^n_+=\left\{(x_1,\dots,x_{n+1})\in S^n\colon x_{n+1}\geq 0\right\}$ is homeomorphic to the n–disk D^n for $n\geq 0$.

Exercise 1.4. Recall that we have considered the set of path components

$$\pi_0(X)=\{\text{path components of } X\}$$

for a topological space X, and constructed a map $\pi_0(f)\colon\pi_0(X)\to\pi_0(Y)$ for every continuous function $f\colon X\to Y$. Show that $\pi_0\colon \mathsf{Top}\to\mathsf{Set}$ is a *functor*; i.e. the following two properties hold:

(1) $\pi_0(f\circ g)=\pi_0(f)\circ\pi_0(g)$ for every $f\colon X\to Y$ and $g\colon Y\to Z$, and every space X,Y,Z.
(2) $\pi_0(\mathrm{id}_X)=\mathrm{id}_{\pi_0(X)}$ for all topological spaces X.